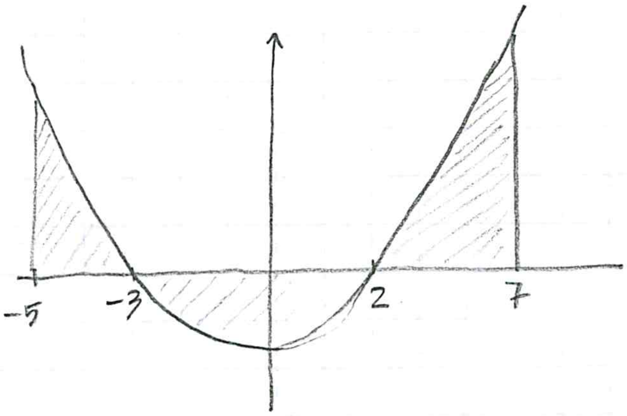
4.4. & 5.5. Indefinite Integrals & the Net Change Theorem
Average Value of a Function.

Ex]: Find the total area b/w the curve $y = f(x) = x^2 + x - 6$ and the x-axis, for x between -5 and 7.

$f(x) = (x+3)(x-2) \Rightarrow$ roots at $x = -3,\ x = 2$



$$\text{Total area} = \int_{-5}^{-3} f(x)\,dx - \int_{-3}^{2} f(x)\,dx + \int_{2}^{7} f(x)$$

$$\int f(x)\,dx = \frac{x^3}{3} + \frac{x^2}{2} - 6x + C$$

$$\Rightarrow \int_{-5}^{-3} f(x)\,dx = \left(\frac{x^3}{3} + \frac{x^2}{2} - 6x\right)\Big|_{-5}^{-3}$$

$$= \left(-\frac{27}{3} + \frac{9}{2} + 18\right) - \left(-\frac{125}{3} + \frac{25}{2} + 30\right) = \frac{38}{3}$$

$$\Rightarrow \int_{-3}^{2} f(x)\,dx = \left(\frac{x^3}{3} + \frac{x^2}{2} - 6x\right)\Big|_{-3}^{2} = \left(\frac{8}{3} + \frac{4}{2} - 12\right) - \left(\frac{-27}{3} + \frac{9}{2} + 18\right) = \frac{-125}{6}$$

$$\Rightarrow \int_{2}^{7} f(x)\,dx = \left(\frac{x^3}{3} + \frac{x^2}{2} - 6x\right)\Big|_{2}^{7} = \left(\frac{343}{3} + \frac{49}{2} - 42\right) - \left(\frac{8}{3} + \frac{4}{2} - 12\right) = \frac{625}{6}$$

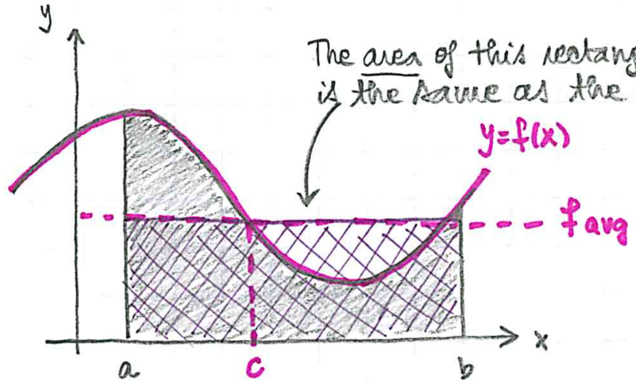
$$\Rightarrow \text{Total area} = \frac{38}{3} + \frac{125}{6} + \frac{625}{6}.$$

$$\int_{-5}^{7} f(x)\,dx = \left(\frac{x^3}{3} + \frac{x^2}{2} - 6x\right)\Big|_{-5}^{7} = \left(\frac{343}{3} + \frac{49}{2} - 42\right) - \left(\frac{-125}{3} + \frac{25}{2} + 30\right) = \frac{576}{6}.$$

Average Value of a function f on the interval $[a,b]$:

$$\boxed{f_{avg} := \frac{1}{b-a}\int_a^b f(x)\,dx}$$

The area of this rectangle, w/ height f_{avg}, is the same as the area under the graph of f from a to b.

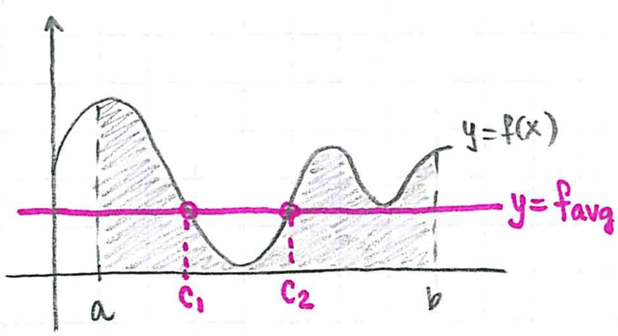


(average height)

Mean Value Theorem for Integrals:

Guarantees that a continuous function attains its average value over $[a,b]$ at least once.

> If f is continuous on $[a,b]$, then there is a number $c \in (a,b)$ such that
>
> $$\boxed{f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx}$$



Proof: Let $F(x) := \int_a^x f(t)\,dt$, $x \in [a,b]$.

by FTC1: F is continuous on $[a,b]$ and differentiable on (a,b), and $F'(x) = f(x)$.

Apply MVT to F: there is $c \in (a,b)$ such that

$$F'(c) = \frac{F(b)-F(a)}{b-a} \quad \left(F(b) = \int_a^b f(x)dx,\ F(a) = \right.$$

$$f(c) = \frac{\int_a^b f(x)\,dx}{b-a} = f_{avg}. \qquad \blacksquare$$

Ex: Average value of $f(x) = 5x^2 - 2x$ on $[2,4]$?

$$f_{avg} = \frac{\int_2^4 f(x)dx}{4-2} = \frac{1}{2}\left(\frac{5x^3}{3} - x^2\right)\Big|_2^4 = \frac{1}{2}\left[\left(\frac{320}{3} - 16\right) - \left(\frac{40}{3} - 4\right)\right]$$

$$= \frac{1}{2}\left(\frac{280}{3} - 12\right) = \frac{140}{3} - 6 = \frac{122}{3}$$

Find $c \in (2,4)$ such that $f(c) = f_{avg}$?

$$5c^2 - 2c = \frac{122}{3} \quad ; \quad 15c^2 - 6c - 122 = 0$$

$$\Delta = 36 + 4 \cdot 15 \cdot 122 = 7356$$

$$c = \frac{6 \pm \sqrt{7356}}{30}$$

$\oplus$ gives $3.06 \in (2,4)$

$\ominus$ gives negative number

$$\Rightarrow \boxed{c = \frac{6 + \sqrt{7356}}{30}}$$

The Net Change Theorem: (just a rephrasing of FTC 2).

Recall: The derivative $f'(x)$ of a function $f(x)$ is the rate of change of f.

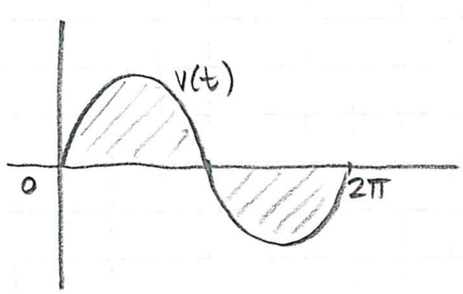
$\Rightarrow$ The net change of a function f over $[a,b]$ is $\int_a^b f'(x)dx = f(b) - f(a)$.
By contrast, the total change of f over $[a,b]$ is $\int_a^b |f'(x)|dx$.
$\hookrightarrow$ counts all changes in f positively.

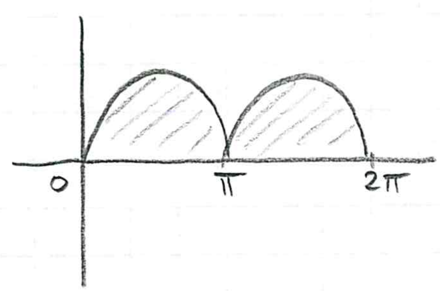
Distance & Displacement:

Displacement = net change in position = integral of velocity. (v)

Distance traveled = total change in position = integral of speed. ($|v|$).

Example: $v(t) = \sin t \quad , \quad 0 \le t \le 2\pi$





Displacement $= \underline{\underline{0}}$

Distance traveled > 0

$$= 2\int_0^{\pi} \sin t \, dt = -2\cos t \Big|_0^{\pi} = -2\cdot(-1) - (-2\cdot 1)$$

$$= 2 + 2 = \underline{\underline{4}}$$

Example: Velocity: $v(t) = t(9-t)$. ; initial position: 2 units to the left of origin.

(a). Position of object 19 s later?

$s(t)$ = position @ time t

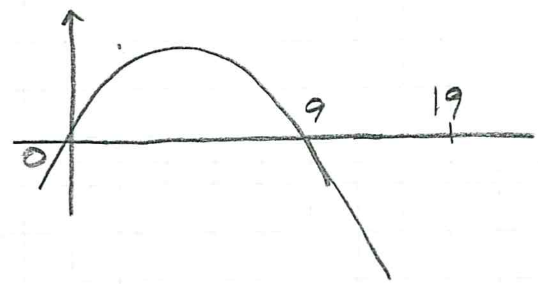
$$= \int v(t)\,dt = \int (9t - t^2)\,dt = \frac{9t^2}{2} - \frac{t^3}{3} + C$$

$$s(0) = -2 \Rightarrow C = -2 \Rightarrow \boxed{s(t) = \frac{9t^2}{2} - \frac{t^3}{3} - 2}$$

$$\Rightarrow s(19) = \frac{9 \cdot 19^2}{2} - \frac{19^3}{3} - 2$$

(b). Total distance traveled in first 19 seconds?

$\hookrightarrow \int_0^{19} |v(t)|\,dt$



$$|v(t)| = \begin{cases} t(9-t), & 0 \leq t \leq 9 \\ -t(9-t), & t > 9 \end{cases}$$

$$\int_0^{19} |v(t)|\,dt = \int_0^9 t(9-t)\,dt + \int_9^{19} -t(9-t)\,dt$$

$$= \int_0^9 (9t - t^2)\,dt + \int_9^{19} (t^2 - 9t)\,dt$$

$$= \left(\frac{9t^2}{2} - \frac{t^3}{3}\right)\Big|_0^9 + \left(\frac{t^3}{3} - \frac{9t^2}{2}\right)\Big|_9^{19} = \frac{5429}{6}$$